§2 场的导数

1

一、场的一阶导数

在直角坐标系下，定义

- 标量场的**梯度**：$\nabla\varphi \triangleq (\partial_i\varphi)\hat{x}_i$
- 矢量场的**梯度**：$\nabla\vec{f} \triangleq (\partial_i f_j)\hat{x}_i\hat{x}_j$

  矢量场的**散度**：$\nabla\cdot\vec{f} \triangleq \partial_i f_i$

  矢量场的**旋度**：$\nabla\times\vec{f} \triangleq (\varepsilon_{ijk}\partial_j f_k)\hat{x}_i$
- 张量场的**散度**：$\nabla\cdot\overleftrightarrow{T} \triangleq (\partial_j T_{ji})\hat{x}_i$

$$\nabla\cdot\overleftrightarrow{T} \triangleq (\hat{x}_i\partial_i)\cdot(T_{jk}\hat{x}_j\hat{x}_k) = (\partial_i T_{jk})(\hat{x}_i\cdot\hat{x}_j\hat{x}_k)$$
$$= (\partial_i T_{jk})(\delta_{ij}\hat{x}_k) = (\partial_i T_{ik})\hat{x}_k$$

2

【例】试求 $r = |\vec{x}|$ 的梯度以及 $\vec{x}$ 的梯度、散度和旋度。

【解】$r = \sqrt{x_k x_k}$ ⟹ $\dfrac{\partial r}{\partial x_i} = \dfrac{1}{2r}\dfrac{\partial(x_k x_k)}{\partial x_i} = \dfrac{2x_k\delta_{ik}}{2r} = \dfrac{2x_i}{2r}$

⟹ $\nabla r = \dfrac{\partial r}{\partial x_i}\hat{x}_i = \dfrac{x_i\hat{x}_i}{r} = \dfrac{\vec{x}}{r}$

⟹ $\nabla r = \hat{r}$

$\vec{x} = x_k\hat{x}_k$ ⟹ $\nabla\vec{x} = (\partial_i x_k)\hat{x}_i\hat{x}_k = \delta_{ik}\hat{x}_i\hat{x}_k = \hat{x}_i\hat{x}_i$

⟹ $\nabla\vec{x} = \overleftrightarrow{I}$

⟹ $\nabla\cdot\vec{x} = \mathrm{tr}\overleftrightarrow{I} = 3,\quad \nabla\times\vec{x} = 0$

3

1. 莱布尼兹法则

$$\nabla(\varphi\psi) = (\nabla\varphi)\psi + \varphi(\nabla\psi)$$

$$\nabla\left(\vec{f}\cdot\vec{g}\right) = \vec{f}\cdot\nabla\vec{g} + \vec{g}\cdot\nabla\vec{f} + \vec{f}\times(\nabla\times\vec{g}) + \vec{g}\times\left(\nabla\times\vec{f}\right)$$

$$\begin{cases}\nabla\left(\varphi\vec{f}\right) = (\nabla\varphi)\vec{f} + \varphi\left(\nabla\vec{f}\right)\\ \nabla\cdot\left(\varphi\vec{f}\right) = (\nabla\varphi)\cdot\vec{f} + \varphi\left(\nabla\cdot\vec{f}\right)\\ \nabla\times\left(\varphi\vec{f}\right) = (\nabla\varphi)\times\vec{f} + \varphi\left(\nabla\times\vec{f}\right)\end{cases}$$

$$\begin{cases}\nabla\left(\vec{f}\times\vec{g}\right) = \left(\nabla\vec{f}\right)\times\vec{g} - (\nabla\vec{g})\times\vec{f}\\ \nabla\cdot\left(\vec{f}\times\vec{g}\right) = \left(\nabla\times\vec{f}\right)\cdot\vec{g} - (\nabla\times\vec{g})\cdot\vec{f}\\ \nabla\times\left(\vec{f}\times\vec{g}\right) = (\nabla\cdot\vec{g} + \vec{g}\cdot\nabla)\vec{f} - \left(\nabla\cdot\vec{f} + \vec{f}\cdot\nabla\right)\vec{g}\end{cases}$$

4

【例】 证明 $\nabla\cdot\left(\vec{f}\vec{g}\right) = \left(\nabla\cdot\vec{f}\right)\vec{g} + \left(\vec{f}\cdot\nabla\right)\vec{g}$。

【证明1】（下标法）：

$$\left[\nabla\cdot\left(\vec{f}\vec{g}\right)\right]_k = \partial_i(f_i g_k) \quad \text{（定义展开）}$$

$$= (\partial_i f_i)g_k + f_i(\partial_i g_k) \quad \text{（莱布尼兹法则）}$$

$$= \left(\nabla\cdot\vec{f}\right)g_k + \left(\vec{f}\cdot\nabla\right)g_k \quad \text{（定义还原）}$$

$$= \left[\left(\nabla\cdot\vec{f}\right)\vec{g} + \left(\vec{f}\cdot\nabla\right)\vec{g}\right]_k \quad \text{（定义还原）}$$

5

或者

$$\nabla\cdot\left(\vec{f}\vec{g}\right) = (\hat{x}_i\partial_i)\cdot(f_j g_k\hat{x}_j\hat{x}_k)$$

$$= [\partial_i(f_j g_k)](\hat{x}_i\cdot\hat{x}_j\hat{x}_k)$$

$$= [(\partial_i f_j)g_k + f_j(\partial_i g_k)]\delta_{ij}\hat{x}_k$$

$$= [(\partial_i f_i)g_k + f_i(\partial_i g_k)]\hat{x}_k$$

$$= \left(\nabla\cdot\vec{f} + \vec{f}\cdot\nabla\right)\vec{g}$$

6

【证明2】（符号法）：

∇_f 只作用于 $\vec{f}$

∇_g 只作用于 $\vec{g}$

$$\nabla\cdot\left(\vec{f}\vec{g}\right)=\nabla_f\cdot\left(\vec{f}\vec{g}\right)+\nabla_g\cdot\left(\vec{f}\vec{g}\right)$$ （莱布尼兹法则）

$$=\nabla_f\cdot\vec{f}\vec{g}+\nabla_g\cdot\vec{f}\vec{g}$$ （就近点乘）

$$=\left(\nabla_f\cdot\vec{f}\right)\vec{g}+\left(\vec{f}\cdot\nabla_g\right)\vec{g}$$ （就近点乘、交换律）

$$=\left(\nabla\cdot\vec{f}+\vec{f}\cdot\nabla\right)\vec{g}$$ （去下标）

7

【例】证明 $\nabla\cdot\left(\vec{f}\vec{g}\vec{h}\right)=\left(\nabla\cdot\vec{f}\right)\vec{g}\vec{h}+\left(\vec{f}\cdot\nabla\vec{g}\right)\vec{h}+\vec{g}\left(\vec{f}\cdot\nabla\vec{h}\right)$。

【证明】（符号法）：

$$\nabla\cdot\left(\vec{f}\vec{g}\vec{h}\right)=\nabla_f\cdot\left(\vec{f}\vec{g}\vec{h}\right)+\nabla_g\cdot\left(\vec{f}\vec{g}\vec{h}\right)+\nabla_h\cdot\left(\vec{f}\vec{g}\vec{h}\right)$$

$$=\nabla_f\cdot\vec{f}\vec{g}\vec{h}+\nabla_g\cdot\vec{f}\vec{g}\vec{h}+\nabla_h\cdot\vec{f}\vec{g}\vec{h}$$

$$=\left(\nabla_f\cdot\vec{f}\right)\vec{g}\vec{h}+\left(\vec{f}\cdot\nabla_g\right)\vec{g}\vec{h}+\left(\vec{f}\cdot\nabla_h\right)\vec{g}\vec{h}$$

$$=\left(\nabla_f\cdot\vec{f}\right)\vec{g}\vec{h}+\left(\vec{f}\cdot\nabla_g\vec{g}\right)\vec{h}+\vec{g}\left(\vec{f}\cdot\nabla_h\vec{h}\right)$$

$$=\left(\nabla\cdot\vec{f}\right)\vec{g}\vec{h}+\left(\vec{f}\cdot\nabla\vec{g}\right)\vec{h}+\vec{g}\left(\vec{f}\cdot\nabla\vec{h}\right)$$

8

【例】证明 $\nabla\times\left(\vec{f}\times\vec{g}\right)=(\nabla\cdot\vec{g}+\vec{g}\cdot\nabla)\vec{f}-\left(\nabla\cdot\vec{f}+\vec{f}\cdot\nabla\right)\vec{g}$。

【证明】（符号法）：

$$\nabla\times\left(\vec{f}\times\vec{g}\right)=\nabla_f\times\left(\vec{f}\times\vec{g}\right)+\nabla_g\times\left(\vec{f}\times\vec{g}\right)$$

$$=\left[(\nabla_f\cdot\vec{g})\vec{f}-\left(\nabla_f\cdot\vec{f}\right)\vec{g}\right]+\left[(\nabla_g\cdot\vec{g})\vec{f}-\left(\nabla_g\cdot\vec{f}\right)\vec{g}\right]$$

$$=\left[(\vec{g}\cdot\nabla_f)\vec{f}-\left(\nabla_f\cdot\vec{f}\right)\vec{g}\right]+\left[(\nabla_g\cdot\vec{g})\vec{f}-\left(\vec{f}\cdot\nabla_g\right)\vec{g}\right]$$

$$=(\nabla\cdot\vec{g}+\vec{g}\cdot\nabla)\vec{f}-\left(\nabla\cdot\vec{f}+\vec{f}\cdot\nabla\right)\vec{g}$$

或者

$$\nabla\times\left(\vec{f}\times\vec{g}\right)=\nabla\cdot\left(\vec{g}\vec{f}-\vec{f}\vec{g}\right)$$

$$=(\nabla\cdot\vec{g}+\vec{g}\cdot\nabla)\vec{f}-\left(\nabla\cdot\vec{f}+\vec{f}\cdot\nabla\right)\vec{g}$$

9

【例】 证明 $\vec{f}\times\left(\nabla\times\vec{f}\right)=\frac{1}{2}\nabla f^2-\vec{f}\cdot\nabla\vec{f}$。

【证明】 由于（习题1.2）

$$\nabla\left(\vec{f}\cdot\vec{g}\right)=\vec{f}\cdot\nabla\vec{g}+\vec{g}\cdot\nabla\vec{f}+\vec{f}\times(\nabla\times\vec{g})+\vec{g}\times\left(\nabla\times\vec{f}\right)$$

令 $\vec{g}=\vec{f}$ 得

$$\nabla f^2=2\vec{f}\cdot\nabla\vec{f}+2\vec{f}\times\left(\nabla\times\vec{f}\right)$$

$$\Rightarrow\quad \vec{f}\times\left(\nabla\times\vec{f}\right)=\frac{1}{2}\nabla f^2-\vec{f}\cdot\nabla\vec{f}$$

或者

$$\vec{f}\times\left(\nabla\times\vec{f}\right)=\frac{1}{2}\nabla\left(\vec{f}\cdot\vec{f}\right)-\vec{f}\cdot\nabla\vec{f}$$

10

【例】 与张量场有关的几个等式。

$$\begin{cases}\nabla\cdot\left(\vec{f}\vec{g}\right)=\left(\nabla\cdot\vec{f}\right)\vec{g}+\left(\vec{f}\cdot\nabla\right)\vec{g}\\ \nabla\cdot\left(\vec{f}\vec{g}\vec{h}\right)=\left(\nabla\cdot\vec{f}\right)\vec{g}\vec{h}+\left(\vec{f}\cdot\nabla\vec{g}\right)\vec{h}+\vec{g}\left(\vec{f}\cdot\nabla\vec{h}\right)\\ \nabla\cdot\left(\varphi\overleftrightarrow{T}\right)=(\nabla\varphi)\cdot\overleftrightarrow{T}+\varphi\left(\nabla\cdot\overleftrightarrow{T}\right)\end{cases}$$

由最后一式可得：

$$\nabla\varphi=\nabla\cdot\left(\varphi\overleftrightarrow{I}\right)$$

11

【例】 证明 $\nabla\cdot\left(\overleftrightarrow{T}\times\vec{x}\right)=-\vec{x}\times\left(\nabla\cdot\overleftrightarrow{T}\right)$，其中 $\overleftrightarrow{T}$ 为对称张量。

【证明】

$$\begin{aligned}\nabla\cdot\left(\overleftrightarrow{T}\times\vec{x}\right)&=(\hat{x}_i\partial_i)\cdot\left[\left(T_{jk}\hat{x}_j\hat{x}_k\right)\times(x_l\hat{x}_l)\right]\\&=\left[\partial_i(T_{jk}x_l)\right]\left(\hat{x}_i\cdot\hat{x}_j\hat{x}_k\times\hat{x}_l\right)\\&=\left[(\partial_iT_{jk})x_l+T_{jk}\delta_{il}\right]\left(\delta_{ij}\hat{x}_k\times\hat{x}_l\right)\\&=(\partial_iT_{ik})x_l(\hat{x}_k\times\hat{x}_l)+T_{lk}(\hat{x}_k\times\hat{x}_l)\\&=\left(\nabla\cdot\overleftrightarrow{T}\right)\times\vec{x}\\&=-\vec{x}\times\left(\nabla\cdot\overleftrightarrow{T}\right)\end{aligned}$$

12

2. 链式法则

设 $t_r = t_r(\vec{x})$，而标量场 $\varphi = \varphi(t_r)$、矢量场 $\vec{A} = \vec{A}(t_r)$，则

$$\begin{cases} \nabla\varphi(t_r) = \dot{\varphi}\nabla t_r \\ \nabla\vec{A}(t_r) = (\nabla t_r)\dot{\vec{A}} \left[\neq \dot{\vec{A}}\nabla t_r\right] \\ \nabla\cdot\vec{A}(t_r) = (\nabla t_r)\cdot\dot{\vec{A}} = \dot{\vec{A}}\cdot\nabla t_r \\ \nabla\times\vec{A}(t_r) = (\nabla t_r)\times\dot{\vec{A}} = -\dot{\vec{A}}\times\nabla t_r \end{cases}$$

其中，$\dot{\varphi} \triangleq \partial\varphi/\partial t_r$，$\dot{\vec{A}} \triangleq \partial\vec{A}/\partial t_r$。

【例】

$$\nabla\varphi(r) = \frac{\partial\varphi}{\partial r}\hat{r},\quad \nabla\times[\varphi(r)\hat{r}] = 0$$

13

三、二阶导数

$$\nabla\cdot(\nabla\varphi) \xleftrightarrow{\nabla\to\vec{h}} \vec{h}\cdot\left(\vec{h}\varphi\right) = h^2\varphi$$

$$\nabla\times(\nabla\varphi) \xleftrightarrow{\nabla\to\vec{h}} \vec{h}\times\left(\vec{h}\varphi\right) = 0$$

$$\nabla\cdot\left(\nabla\times\vec{F}\right) \xleftrightarrow{\nabla\to\vec{h}} \vec{h}\cdot\left(\vec{h}\times\vec{F}\right) = 0$$

$$\nabla\times\left(\nabla\times\vec{F}\right) \xleftrightarrow{\nabla\to\vec{h}} \vec{h}\times\left(\vec{h}\times\vec{F}\right) = \left(\vec{h}\cdot\vec{F}\right)\vec{h} - h^2\vec{F}$$

$$= \vec{h}\left(\vec{h}\cdot\vec{F}\right) - h^2\vec{F}$$

$$\nabla\left(\nabla\cdot\vec{F}\right)$$

14

1. 两个恒等式

【两个恒等式】 $\nabla\times\nabla\varphi \equiv 0,\quad \nabla\cdot\left(\nabla\times\vec{F}\right) \equiv 0$

【定理】如果 $\nabla\times\vec{F} = 0$，那么存在 φ，使得 $\vec{F} = \nabla\varphi$。

【定理】如果 $\nabla\cdot\vec{F} = 0$， 那么存在 $\vec{A}$，使得 $\vec{F} = \nabla\times\vec{A}$。

$$(\nabla\times\nabla\varphi)_x = \partial_y(\nabla\varphi)_z - \partial_z(\nabla\varphi)_y = \partial_y\partial_z\varphi - \partial_z\partial_y\varphi = 0$$

$$\nabla\cdot\left(\nabla\times\vec{F}\right) = \partial_x\left(\nabla\times\vec{F}\right)_x + \partial_y\left(\nabla\times\vec{F}\right)_y + \partial_z\left(\nabla\times\vec{F}\right)_z$$

$$= \partial_x\left(\partial_yF_z - \partial_zF_y\right) + \partial_y\left(\partial_zF_x - \partial_xF_z\right) + \partial_z\left(\partial_xF_y - \partial_yF_x\right)$$

$$= \left(\partial_x\partial_yF_z - \partial_x\partial_zF_y\right) + \left(\partial_y\partial_zF_x - \partial_y\partial_xF_z\right) + \left(\partial_z\partial_xF_y - \partial_z\partial_yF_x\right)$$

$$= 0$$

15

2. 拉普拉斯算子

梯度场 $\nabla\varphi$ 的散度为

$$\nabla^2\varphi = \nabla\cdot\nabla\varphi = \frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}$$

其中，∇^2 称为**拉普拉斯算子**：

$$\nabla^2 \triangleq \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

拉普拉斯算子也可作用于矢量场上，得到另一个矢量场：

$$\nabla^2\vec{F} \triangleq (\nabla^2 F_x)\hat{x}+(\nabla^2 F_y)\hat{y}+(\nabla^2 F_z)\hat{z} = \frac{\partial^2\vec{F}}{\partial x^2}+\frac{\partial^2\vec{F}}{\partial y^2}+\frac{\partial^2\vec{F}}{\partial z^2}$$

- 并算子

$$\nabla\nabla \triangleq \hat{x}_i\hat{x}_j\partial_i\partial_j$$

16

3. 另一个常用的恒等式

$$\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F})-\nabla^2\vec{F}$$

$$[\nabla\times(\nabla\times\vec{F})]_x = \frac{\partial}{\partial y}(\nabla\times\vec{F})_z-\frac{\partial}{\partial z}(\nabla\times\vec{F})_y$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}F_y-\frac{\partial}{\partial y}F_x\right)-\frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}F_x-\frac{\partial}{\partial x}F_z\right)$$

$$= \left(\frac{\partial^2}{\partial x\partial y}F_y-\frac{\partial^2}{\partial y^2}F_x\right)-\left(\frac{\partial^2}{\partial z^2}F_x-\frac{\partial^2}{\partial x\partial z}F_z\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}F_x+\frac{\partial}{\partial y}F_y+\frac{\partial}{\partial z}F_z\right)-\left(\frac{\partial^2}{\partial x^2}F_x+\frac{\partial^2}{\partial y^2}F_x+\frac{\partial^2}{\partial z^2}F_x\right)$$

$$= \frac{\partial}{\partial x}(\nabla\cdot\vec{F})-\nabla^2 F_x$$

17

四、泰勒展开

1. 单变量函数的泰勒展开

$$\varphi(x+\epsilon) = \varphi(x)+\epsilon\frac{d\varphi(x)}{dx}+\epsilon^2\frac{1}{2!}\frac{d^2\varphi(x)}{dx^2}+\cdots$$

$$= \left[1+\epsilon\frac{d}{dx}+\epsilon^2\frac{1}{2!}\frac{d^2}{dx^2}+\cdots\right]\varphi(x)$$

$$= \left[\sum_{n=0}^{\infty}\frac{\epsilon^n}{n!}\frac{d^n}{dx^n}\right]\varphi(x)$$

$$\Longrightarrow \quad \varphi(x+\epsilon) = \exp\left(\epsilon\frac{d}{dx}\right)\varphi(x)$$

18

2. 标量场的泰勒展开

$$\varphi(\vec{x}+\vec{\epsilon}) = e^{\epsilon_1\partial_1}e^{\epsilon_2\partial_2}e^{\epsilon_3\partial_3}\varphi(\vec{x})$$

$$= e^{\vec{\epsilon}\cdot\nabla}\varphi(\vec{x})$$

$$= \left[1+\vec{\epsilon}\cdot\nabla+\frac{1}{2!}(\vec{\epsilon}\cdot\nabla)^2+\cdots\right]\varphi(\vec{x})$$

$$\Rightarrow \quad \varphi(\vec{x}+\vec{\epsilon}) = e^{\vec{\epsilon}\cdot\nabla}\varphi(\vec{x})$$

$$= \left[1+\vec{\epsilon}\cdot\nabla+\frac{1}{2!}(\vec{\epsilon}\cdot\nabla)^2+\cdots\right]\varphi(\vec{x})$$

$$= \left[1+\vec{\epsilon}\cdot\nabla+\frac{1}{2!}(\vec{\epsilon}\vec{\epsilon}:\nabla\nabla)+\cdots\right]\varphi(\vec{x})$$

19

五、相对位矢

场点：$\vec{x} = x_i\hat{x}_i \quad \longrightarrow \quad \nabla = \hat{x}_i(\partial/\partial x_i)$

源点：$\vec{x}' = x_i'\hat{x}_i \quad \longrightarrow \quad \nabla' = \hat{x}_i(\partial/\partial x_i')$

场点相对于源点的位矢：$\vec{\mathbb{R}} \triangleq \vec{x}-\vec{x}'$

- 相对位矢满足的基本关系：

$$\begin{cases} \nabla\mathbb{R} = \hat{\mathbb{R}} = -\nabla'\mathbb{R} \\ \nabla\vec{\mathbb{R}} = \overleftrightarrow{I} = -\nabla'\vec{\mathbb{R}} \\ \nabla\cdot\vec{\mathbb{R}} = 3 = -\nabla'\cdot\vec{\mathbb{R}} \\ \nabla\times\vec{\mathbb{R}} = 0 = \nabla'\times\vec{\mathbb{R}} \end{cases}$$

20

- 与相对位矢有关的几个常用关系：

（1） $\nabla\varphi(\vec{\mathbb{R}}) = -\nabla'\varphi(\vec{\mathbb{R}}),\quad \nabla\varphi(\mathbb{R}) = \varphi'(\mathbb{R})\hat{\mathbb{R}}.$

（2） $$\begin{cases} \nabla\vec{A}(\mathbb{R}) = (\nabla\mathbb{R})\vec{A}' = \hat{\mathbb{R}}\vec{A}' \\ \nabla\cdot\vec{A}(\mathbb{R}) = (\nabla\mathbb{R})\cdot\vec{A}' = \hat{\mathbb{R}}\cdot\vec{A}' \\ \nabla\times\vec{A}(\mathbb{R}) = (\nabla\mathbb{R})\times\vec{A}' = \hat{\mathbb{R}}\times\vec{A}' \end{cases}$$

（3） $$\frac{1}{\mathbb{R}} = \frac{1}{|\vec{x}-\vec{x}'|} = e^{-\vec{x}'\cdot\nabla}\frac{1}{r}$$

$$= \left[1-\vec{x}'\cdot\nabla+\frac{1}{2!}(\vec{x}'\vec{x}':\nabla\nabla)+\cdots\right]\frac{1}{r}$$

21

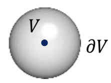
六、与坐标系无关的定义

与坐标系无关的梯度定义：

$$d\varphi(\vec{r}) = \nabla\varphi\cdot d\vec{l}$$

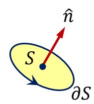
与坐标系无关的散度定义（闭曲面以外法向为正方向）：

$$\nabla\cdot\vec{F} \triangleq \lim_{V\to 0}\left[\frac{1}{V}\oiint_{\partial V}\vec{F}\cdot d\vec{S}\right]$$



与坐标系无关的旋度定义（$\hat{n}$ 为 S 的法向，开曲面的法向 $\hat{n}$ 与其边界绕行方向满足右手法则）：

$$\hat{n}\cdot(\nabla\times\vec{F}) \triangleq \lim_{S\to 0}\left[\frac{1}{S}\oint_{\partial S}\vec{F}\cdot d\vec{l}\right]$$



22

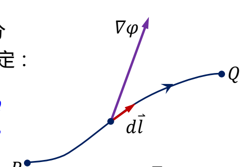
§3 场积分的基本定理

23

一、梯度积分的基本定理

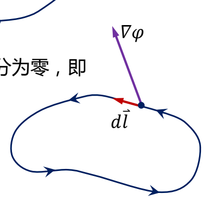
标量场的梯度在曲线 L 上的积分由其在边界 $\partial L=\{1,2\}$ 上的数值决定：

$$\int_P^Q d\vec{l}\cdot\nabla\varphi = \varphi(Q)-\varphi(P) = \varphi\Big|_P^Q$$



【推论】 标量场的梯度在闭曲线上的积分为零，即

$$\oint_C d\vec{l}\cdot\nabla\varphi = 0$$



24

1. 算子等式

$$\int_P^Q d\vec{l}\cdot\nabla\square = \square\Big|_P^Q$$

- 取$\varphi=\vec{F}\cdot\vec{c}$，其中$\vec{c}$为任一常矢量：

$$\int_P^Q d\vec{l}\cdot\nabla(\vec{F}\cdot\vec{c}) = (\vec{F}\cdot\vec{c})\Big|_P^Q \Longrightarrow \left[\int_P^Q d\vec{l}\cdot\nabla\vec{F}\right]\cdot\vec{c} = \left[\vec{F}\Big|_P^Q\right]\cdot\vec{c}$$

- 取$\varphi=\vec{c}\cdot\overleftrightarrow{T}\cdot\vec{d}$，其中$\vec{c}$、$\vec{d}$ 为任两常矢量：

$$\int_P^Q d\vec{l}\cdot\nabla\left(\vec{c}\cdot\overleftrightarrow{T}\cdot\vec{d}\right) = \left(\vec{c}\cdot\overleftrightarrow{T}\cdot\vec{d}\right)\Big|_P^Q$$

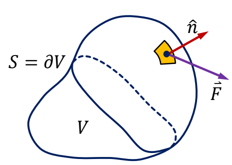
$$\Longrightarrow \vec{c}\cdot\left[\int_P^Q d\vec{l}\cdot\nabla\overleftrightarrow{T}\right]\cdot\vec{d} = \vec{c}\cdot\left[\overleftrightarrow{T}\Big|_P^Q\right]\cdot\vec{d}$$

25

二、散度积分的基本定理

矢量场的散度在区域V 内的积分
由其在边界$S=\partial V$上的数值决定（Gauss定理）

$$\int_V dV\,\nabla\cdot\vec{F} = \oint_{\partial V} d\vec{\sigma}\cdot\vec{F}$$



26

1. 算子等式

$$\int_V dV\,\nabla\square = \oint_{\partial V} d\vec{\sigma}\,\square$$

设$\vec{c}$为任一给定常矢量：

$$\begin{cases} \vec{F}=\varphi\vec{c} \Longrightarrow \int_V dV\,\nabla\varphi = \oint_{\partial V} d\vec{\sigma}\,\varphi \\ \vec{F}=\vec{A}\times\vec{c} \Longrightarrow \int_V dV\,\nabla\times\vec{A} = \oint_{\partial V} d\vec{\sigma}\times\vec{A} \\ \vec{F}=\overleftrightarrow{T}\cdot\vec{c} \Longrightarrow \int_V dV\,\nabla\cdot\overleftrightarrow{T} = \oint_{\partial V} d\vec{\sigma}\cdot\overleftrightarrow{T} \end{cases}$$

27

2. 两个推论

【推论1】闭曲面的面积矢量为零：

$$\oint_{\partial V} d\vec{\sigma} = 0$$

【推论2】梯度、散度、旋度的另一与坐标系无关的定义：

$$\nabla \square = \lim_{V\to 0}\frac{1}{V}\oint_{\partial V} d\vec{\sigma}\,\square \quad \Longrightarrow \quad \begin{cases} \nabla\varphi \triangleq \lim\limits_{V\to 0}\dfrac{1}{V}\oint_{\partial V} d\vec{\sigma}\,\varphi \\ \nabla\cdot\vec{F} \triangleq \lim\limits_{V\to 0}\dfrac{1}{V}\oint_{\partial V} d\vec{\sigma}\cdot\vec{F} \\ \nabla\times\vec{F} \triangleq \lim\limits_{V\to 0}\dfrac{1}{V}\oint_{\partial V} d\vec{\sigma}\times\vec{F} \end{cases}$$

28

3. 格林公式

在Gauss定理中：

- 取 $\vec{F} = \varphi\nabla\psi$，得到**格林第一公式：**

$$\int_V dV\,[\varphi\nabla^2\psi + \nabla\varphi\cdot\nabla\psi] = \oint_{\partial V} d\vec{\sigma}\cdot\varphi\nabla\psi$$

- 取 $\vec{F} = \varphi\nabla\psi - \psi\nabla\varphi$，得到**格林第二公式：**

$$\int_V dV\,[\varphi\nabla^2\psi - \psi\nabla^2\varphi] = \oint_{\partial V} d\vec{\sigma}\cdot[\varphi\nabla\psi - \psi\nabla\varphi]$$

- 取 $\vec{F} = \varphi\nabla\varphi$，得到**格林第三公式：**

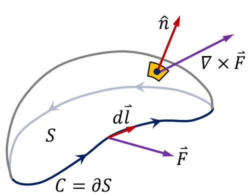
$$\int_V dV\,[\varphi\nabla^2\varphi + (\nabla\varphi)^2] = \oint_{\partial V} d\vec{\sigma}\cdot\varphi\nabla\varphi = \oint_{\partial V}\varphi\frac{\partial\varphi}{\partial n}d\sigma$$

29

三、旋度积分的基本定理

矢量场的旋度在区域 Σ 内的积分
由其在边界 $C = \partial\Sigma$上的数值决定（Stokes定理）

$$\oint_{\Sigma} d\vec{\sigma}\cdot(\nabla\times\vec{F}) = \oint_{\partial\Sigma} d\vec{l}\cdot\vec{F} \quad \leftrightarrow \quad \oint_{\Sigma}(d\vec{\sigma}\times\nabla)\cdot\vec{F} = \oint_{\partial\Sigma} d\vec{l}\cdot\vec{F}$$



30

1. 算子等式

$$\oint_{\Sigma}(d\vec{\sigma}\times\nabla)\,\square = \oint_{\partial\Sigma} d\vec{l}\,\square$$

设 $\vec{c}$ 为任一给定常矢量：

$$\begin{cases}\vec{F}=\varphi\vec{c} \Longrightarrow \oint_{\Sigma} d\vec{\sigma}\times\nabla\varphi = \oint_{\partial\Sigma} d\vec{l}\,\varphi \\ \vec{F}=\vec{A}\times\vec{c} \Longrightarrow \oint_{\Sigma}(d\vec{\sigma}\times\nabla)\times\vec{A} = \oint_{\partial\Sigma} d\vec{l}\times\vec{A} \\ \vec{F}=\overleftrightarrow{T}\cdot\vec{c} \Longrightarrow \oint_{\Sigma}(d\vec{\sigma}\times\nabla)\cdot\overleftrightarrow{T} = \oint_{\partial\Sigma} d\vec{l}\cdot\overleftrightarrow{T}\end{cases}$$

31

§5 正交曲线坐标系

（本节不采用求和约定）

32

一、正交曲线坐标系

1. 曲线坐标系 (u_1,u_2,u_3) ：

$$\vec{x}=\vec{x}(u_1,u_2,u_3) \quad\Longrightarrow\quad d\vec{x}=d\vec{l}=\sum_{a=1}^{3}\frac{\partial\vec{x}}{\partial u_a}du_a=\sum_{a=1}^{3}(h_a du_a)\hat{u}_a$$

- **拉梅系数：** $h_a \triangleq \left|\frac{\partial\vec{x}}{\partial u_a}\right|$
- **基矢量：** $\hat{u}_a \triangleq \frac{\partial\vec{x}/\partial u_a}{|\partial\vec{x}/\partial u_a|}=\frac{1}{h_a}\frac{\partial\vec{x}}{\partial u_a}$
- **体积因子：** $H \triangleq \frac{\partial\vec{x}}{\partial u_1}\cdot\left(\frac{\partial\vec{x}}{\partial u_2}\times\frac{\partial\vec{x}}{\partial u_3}\right)=h_1h_2h_3\hat{u}_1\cdot(\hat{u}_2\times\hat{u}_3)$

33

2. 正交曲线坐标系

$$\hat{u}_a \cdot \hat{u}_b = \delta_{ab}$$

- 不妨设为右手系：

$$\hat{u}_a \times \hat{u}_b = \sum_{c=1}^{3} \varepsilon_{abc} \hat{u}_c$$

- 正交曲线坐标系下的体积因子为：

$$H = h_1 h_2 h_3$$

34

【例】试分别写出球坐标系和柱坐标系的Lame系数。

【解】利用球坐标系的定义

$$\vec{x} = (r \sin\theta \cos\phi)\hat{x}_1 + (r \sin\theta \sin\phi)\hat{x}_2 + (r \cos\theta)\hat{x}_3$$

$$\Rightarrow \begin{cases} \dfrac{\partial \vec{x}}{\partial r} = (\sin\theta \cos\phi)\hat{x}_1 + (\sin\theta \sin\phi)\hat{x}_2 + (\cos\theta)\hat{x}_3 \\ \dfrac{\partial \vec{x}}{\partial \theta} = (r \cos\theta \cos\phi)\hat{x}_1 + (r \cos\theta \sin\phi)\hat{x}_2 + (-r \sin\theta)\hat{x}_3 \\ \dfrac{\partial \vec{x}}{\partial \phi} = (-r \sin\theta \sin\phi)\hat{x}_1 + (r \sin\theta \cos\phi)\hat{x}_2 \end{cases}$$

$$\Rightarrow h_r = \left|\frac{\partial \vec{x}}{\partial r}\right| = 1, \quad h_\theta = \left|\frac{\partial \vec{x}}{\partial \theta}\right| = r, \quad h_\phi = \left|\frac{\partial \vec{x}}{\partial \phi}\right| = r \sin\theta$$

35

或者，由于$d\vec{x} = (dr)\hat{r} + (rd\theta)\hat{\theta} + (r \sin\theta \, d\phi)\hat{\phi}$，因而

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta$$

而体积因子

$$H \triangleq h_r h_\theta h_\phi = r^2 \sin\theta$$

类似地，柱坐标系中$d\vec{x} = (ds)\hat{s} + (sd\phi)\hat{\phi} + (dz)\hat{z}$，因而

$$h_s = 1, \quad h_\phi = s, \quad h_z = 1.$$

而体积因子

$$H \triangleq h_s h_\phi h_z = s$$

36

二、场的导数

1. 正交曲线坐标系下的梯度算子

由于

$$\begin{cases} d\varphi = \sum_{b=1}^{3} \frac{\partial \varphi}{\partial u_b} du_b \\ d\varphi = \nabla\varphi \cdot d\vec{l} = \sum_{b=1}^{3} (\nabla\varphi)_b (h_b du_b) \end{cases}$$

因而，φ 的梯度在曲线坐标系下的第 b 个分量为：

$$(\nabla\varphi)_b = \frac{1}{h_b}\frac{\partial \varphi}{\partial u_b} \longrightarrow \nabla\varphi = \sum_{b=1}^{3}\left(\frac{1}{h_b}\frac{\partial \varphi}{\partial u_b}\right)\hat{u}_b$$

37

在正交曲线坐标系下，梯度算子表示为

$$\nabla = \sum_{b=1}^{3} \frac{\hat{u}_b}{h_b}\frac{\partial}{\partial u_b}$$

- 球坐标系：$\nabla\varphi = \frac{\partial \varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial \varphi}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial \varphi}{\partial \phi}\hat{\phi}$
- 柱坐标系：$\nabla\varphi = \frac{\partial \varphi}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial \varphi}{\partial \phi}\hat{\phi} + \frac{\partial \varphi}{\partial z}\hat{z}$
- 特例：$\nabla u_a = \frac{\hat{u}_a}{h_a}$，$\nabla f(u_a) = \frac{\hat{u}_a}{h_a}\frac{\partial f}{\partial u_a} \parallel \hat{u}_a$

38

【例】试在正交曲线坐标系下写出矢量场 $\vec{F}(u_1, u_2, u_3)$的散度。

【解】

$$\begin{aligned} \nabla\cdot\vec{F} &= \sum_{a=1}^{3}\frac{\hat{u}_a}{h_a}\frac{\partial}{\partial u_a}\cdot\sum_{b=1}^{3}F_b\hat{u}_b \\ &= \sum_{a,b=1}^{3}\frac{\hat{u}_a}{h_a}\cdot\frac{\partial(F_b\hat{u}_b)}{\partial u_a} \\ &= \sum_{a,b=1}^{3}\frac{\hat{u}_a\cdot\hat{u}_b}{h_a}\frac{\partial F_b}{\partial u_a} + \sum_{a,b=1}^{3}\frac{\hat{u}_a}{h_a}\cdot F_b\frac{\partial \hat{u}_b}{\partial u_a} \\ &= \sum_{a=1}^{3}\frac{1}{h_a}\frac{\partial F_a}{\partial u_a} + \sum_{a,b=1}^{3}F_b\frac{\hat{u}_a}{h_a}\cdot\frac{\partial \hat{u}_b}{\partial u_a} \end{aligned}$$

39

或者，利用莱布尼茨法则有

$$\nabla\cdot\vec{F}=\sum_{a=1}^{3}\nabla\cdot(F_a\hat{u}_a)=\sum_{a=1}^{3}(\nabla F_a)\cdot\hat{u}_a+\sum_{a=1}^{3}F_a(\nabla\cdot\hat{u}_a)$$

其中

$$\sum_{a=1}^{3}(\nabla F_a)\cdot\hat{u}_a=\sum_{a=1}^{3}\left[\sum_{b=1}^{3}\frac{\hat{u}_b}{h_b}\frac{\partial F_a}{\partial u_b}\right]\cdot\hat{u}_a=\sum_{a=1}^{3}\frac{1}{h_a}\frac{\partial F_a}{\partial u_a}$$

但是

$$\nabla\cdot\hat{u}_a=?$$

40

利用坐标的梯度$\nabla u_a=\hat{u}_a/h_a$有

$$\nabla\times\nabla u_a=0 \Longrightarrow \nabla\times\frac{\hat{u}_a}{h_a}=0$$

又由于

$$\begin{cases}\nabla u_1\times\nabla u_2=\dfrac{\hat{u}_1}{h_1}\times\dfrac{\hat{u}_2}{h_2}=\dfrac{h_3\hat{u}_3}{H}\\ \nabla u_1\times\nabla u_2=\nabla\times(u_1\nabla u_2)-u_1(\nabla\times\nabla u_2)\end{cases}$$

$$\Longrightarrow \nabla\cdot(\nabla u_1\times\nabla u_2)=\nabla\cdot[\nabla\times(u_1\nabla u_2)]=0$$

$$\Longrightarrow \nabla\cdot\frac{h_3\hat{u}_3}{H}=0 \Longrightarrow \nabla\cdot\frac{h_a\hat{u}_a}{H}=0$$

41

2. 矢量分析三剑客

$$\nabla u_a=\frac{\hat{u}_a}{h_a},\quad \nabla\times\frac{\hat{u}_a}{h_a}=0,\quad \nabla\cdot\frac{h_a\hat{u}_a}{H}=0$$

- 倘若将三剑客与莱布尼茨法则结合使用，就形成了三位欧几里德空间中矢量分析的必杀技，足以秒杀理论物理学习和研究中所遭遇的所有矢量分析问题。

$$\nabla\cdot(\varphi\vec{F})=(\nabla\varphi)\cdot\vec{F}+\varphi(\nabla\cdot\vec{F})$$
$$\nabla\times(\varphi\vec{F})=(\nabla\varphi)\times\vec{F}+\varphi(\nabla\times\vec{F})$$

- 另一个关系则可为分析带来进一步的便利：

$$\nabla f(u_a)=\frac{\hat{u}_a}{h_a}\frac{\partial f}{\partial u_a}\parallel\hat{u}_a$$

42

【例】 已知矢量场在柱坐标系中的表达式为 $\vec{F}=\hat{\phi}\ln s$ 。试求其散度和旋度。

【解】 由于柱坐标系的Lame系数为：$h_s=1,\quad h_\phi=s,\quad h_z=1.$

所以存在恒等式： $\nabla\cdot\hat{\phi}=0,\quad \nabla\times\frac{\hat{\phi}}{s}=0$

散度: $\nabla\cdot(\hat{\phi}\ln s)=(\nabla\cdot\hat{\phi})\ln s+\hat{\phi}\cdot(\nabla\ln s)=\hat{\phi}\cdot\frac{\hat{s}}{s}=0$

旋度: $\nabla\times(\hat{\phi}\ln s)=\nabla\times\left(\frac{\hat{\phi}}{s}s\ln s\right)$

$$=\left(\nabla\times\frac{\hat{\phi}}{s}\right)(s\ln s)+\nabla(s\ln s)\times\frac{\hat{\phi}}{s}$$

$$=\frac{\partial(s\ln s)}{\partial s}\hat{s}\times\frac{\hat{\phi}}{s}\quad=\frac{\ln s+1}{s}\hat{z}$$

43

【思考】 流体力学的Navier-Stokes 方程涉及流速 $\vec{v}$ 与其旋度矢量积 $\vec{A}=\vec{v}\times(\nabla\times\vec{v})$ 的旋度。已知某流体的流速矢量在柱坐标系中表为 $\vec{v}=\hat{z}\ln s$。 请针对这样的流速计算 $\vec{A}$ 的旋度和散度。

44

3. 正交曲线坐标系下的散度

$$\nabla\cdot\vec{F}=\sum_a\nabla\cdot(\hat{u}_aF_a)=\sum_a\nabla\cdot\left(\frac{h_a\hat{u}_a}{H}\frac{HF_a}{h_a}\right)$$

$$=\sum_a\frac{h_a\hat{u}_a}{H}\cdot\left(\nabla\frac{HF_a}{h_a}\right)$$

$$=\sum_a\frac{h_a\hat{u}_a}{H}\cdot\left[\sum_b\frac{\hat{u}_b}{h_b}\frac{\partial}{\partial u_b}\left(\frac{HF_a}{h_a}\right)\right]$$

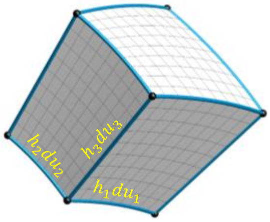
$$=\sum_{a,b}\frac{h_a\hat{u}_a\cdot\hat{u}_b}{Hh_b}\frac{\partial}{\partial u_b}\left(\frac{HF_a}{h_a}\right)$$

$$=\sum_{a,b}\frac{h_a\delta_{ab}}{Hh_b}\frac{\partial}{\partial u_b}\left(\frac{HF_a}{h_a}\right)$$

45

$$\nabla\cdot\vec{F}=\frac{1}{H}\sum_a\frac{\partial}{\partial u_a}\left(\frac{HF_a}{h_a}\right)$$

$$=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3F_1)+\frac{\partial}{\partial u_2}(h_3h_1F_2)+\frac{\partial}{\partial u_3}(h_1h_2F_3)\right]$$



46

- 球坐标系： $h_r=1$, $h_\theta=r$, $h_\phi=r\sin\theta$， $H=r^2\sin\theta$

$$\nabla\cdot\vec{F}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta)+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

- 柱坐标系： $h_s=1$, $h_\phi=s$, $h_z=1$, $H=s$

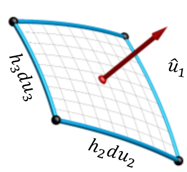
$$\nabla\cdot\vec{F}=\frac{1}{s}\frac{\partial}{\partial s}(sF_s)+\frac{1}{s}\frac{\partial F_\phi}{\partial\phi}+\frac{\partial F_z}{\partial z}$$

47

【思考】证明：在正交曲线坐标系中，矢量场$\vec{F}(\vec{r})$的旋度写为

$$\nabla\times\vec{F}=\frac{1}{H}\begin{vmatrix}h_1\hat{u}_1 & h_2\hat{u}_2 & h_3\hat{u}_3\\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3\\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix}$$

由此分别给出$\nabla\times\vec{F}$在球坐标系和柱坐标系中的表达式。



48

4. 正交曲线坐标系下的拉普拉斯

利用前面给出的散度表达式

$$\nabla\cdot\vec{F}=\frac{1}{H}\sum_a\frac{\partial}{\partial u_a}\left(\frac{HF_a}{h_a}\right)$$

由于 $\nabla^2\varphi=\nabla\cdot\nabla\varphi$，在上式中令

$$F_a=(\nabla\varphi)_a=\frac{1}{h_a}\frac{\partial\varphi}{\partial u_a}$$

因此

$$\nabla^2\varphi=\frac{1}{H}\sum_a\frac{\partial}{\partial u_a}\left(\frac{H}{h_a^2}\frac{\partial\varphi}{\partial u_a}\right)$$

49

$$\nabla^2\varphi=\frac{1}{H}\sum_a\frac{\partial}{\partial u_a}\left(\frac{H}{h_a^2}\frac{\partial\varphi}{\partial u_a}\right)$$

$$=\frac{1}{H}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\varphi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_2h_3}{h_2}\frac{\partial\varphi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\varphi}{\partial u_1}\right)\right]$$

- 球坐标系： $h_r=1,\quad h_\theta=r,\quad h_\phi=r\sin\theta,\quad H=r^2\sin\theta\,.$

$$\nabla^2\varphi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right)=\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varphi)$$

- 柱坐标系： $h_s=1,\quad h_\phi=s,\quad h_z=1,\quad H=s$

$$\nabla^2\varphi=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\varphi}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2\varphi}{\partial\phi^2}+\frac{\partial^2\varphi}{\partial z^2}$$

50

【例】试计算 $\hat{r}/r^2$ 的旋度和散度。

【解】球坐标系中存在恒等式：$\nabla\times\hat{r}=0,\quad \nabla\cdot\frac{\hat{r}}{r^2\sin\theta}=0$

（1）旋度【方法一：微分】

$$\nabla\times\frac{\hat{r}}{r^2}=\nabla\frac{1}{r^2}\times\hat{r}\;=-\frac{2\nabla r}{r^3}\times\hat{r}\;=-\frac{2}{r^3}\hat{r}\times\hat{r}$$

$$\Rightarrow\quad\nabla\times\frac{\hat{r}}{r^2}=0$$

（此结果也可利用前面给出的球坐标系下的旋度公式得到）

51

【方法二：积分】利用Stokes定理，有

$$\oint_\Sigma d\vec{\sigma}\cdot\left(\nabla\times\frac{\hat{r}}{r^2}\right)=\oint_{\partial\Sigma} d\vec{l}\cdot\frac{\hat{r}}{r^2}=\oint_{\partial\Sigma}\frac{dr}{r^2}=\oint_{\partial\Sigma} d\left(-\frac{1}{r}\right)=0$$

由于此结论对任一给定的曲面Σ都成立，因而

$$\nabla\times\frac{\hat{r}}{r^2}=0$$

积分方法和微分方法给出的结果相同。

或者，对于任一给定区域 V ，有

$$\int_V dV\,\nabla\times\frac{\hat{r}}{r^2}=\oint_{\partial V} d\vec{\sigma}\times\frac{\hat{r}}{r^2}=-\oint_{\partial V} d\vec{\sigma}\times\nabla\frac{1}{r}=-\oint_{\partial\partial V}\frac{d\vec{l}}{r}=0$$

52

（2）散度【方法一：微分】

$$\nabla\cdot\frac{\hat{r}}{r^2}=\nabla\cdot\left(\frac{\hat{r}}{r^2\sin\theta}\sin\theta\right)=\frac{\hat{r}}{r^2\sin\theta}\cdot(\nabla\sin\theta)$$

$$=\frac{\hat{r}}{r^2\sin\theta}\cdot\frac{\hat{\theta}}{r}\frac{\partial\sin\theta}{\partial\theta}=\hat{r}\cdot\hat{\theta}\frac{\cot\theta}{r^3}=0$$

或者直接利用前面给出的球坐标系下的散度公式：

$$\nabla\cdot\vec{F}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta)+\frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\Longrightarrow\ \nabla\cdot\frac{\hat{r}}{r^2}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\cdot\frac{1}{r^2}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}(1)=0$$

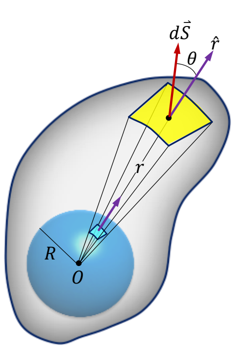
53

【方法二：积分】利用高斯定理，
对任一给定的三维区域 V 有

$$\int_V dV\,\nabla\cdot\frac{\hat{r}}{r^2}=\oint_{\partial V} d\vec{\sigma}\cdot\frac{\hat{r}}{r^2}$$

$$=\oint_{\partial V} d\Omega$$

$$=\begin{cases}4\pi, & O\in V\\ 0, & O\notin V\end{cases}$$



可见 $\hat{r}/r^2$ 的散度不为零！

微分方法和积分方法给出的结论不一致!

54

谢谢同学们！

55